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Complexity of the Object Allocation Problem with Minimum Number of Changes*

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Abstract

This paper studies the problem of reallocating objects to agents while taking into account agents' endowments, object capacities and agents' preferences. The goal is to find a Pareto efficient and individually rational allocation that minimizes the number of individuals who need to change from their initial allocation to the final one. We call this problem as MINDIST. We establish NP-completeness result for MINDIST. We also show that MINDIST remains NP-complete when we restrict individual preferences to be binary, meaning that each individual can rank at most two objects in the preferences. Finally, we present an integer programming formulation to solve small to moderately sized instances of the NP-hard problems.

JEL codes: C78

Keywords: Object allocation, Pareto-efficiency, Individual Rationality, Computational complexity, Minimum changes

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1 Introduction

Many real-world situations require reallocating objects to agents in the absence of a price-setting mechanism, or when such a mechanism is undesirable. Reallocating objects can also impose significant costs on the central authority. For instance, in the case of reallocating tenants in social housing—typically provided to those facing economic hardship—relocation can be logistically expensive and disruptive to tenants' daily lives.[1] When reallocating tenants, housing authorities must consider the costs associated with such moves. To minimize these costs, an authority may aim to find an allocation that is both Pareto efficient and individually rational, while minimizing the number of tenants required to move.[2]

Another relevant example is the reassignment of experienced public employees across public institutions, such as judges or prosecutors to courts in Italy. Certain disadvantaged regions experience high turnover among skilled public servants. To address this, the Ministry of Justice may aim to identify an allocation that minimizes the number of judges requiring relocation, selecting from among allocations that are Pareto efficient and individually rational.[3]

These considerations motivate the present study, which focuses on the problem of finding a Pareto efficient and individually rational allocation that minimizes the number of agents who are reallocated from their initial endowments. We call this problem as MINDIST.[4]

First, we address MINDIST and demonstrate that it is computationally challenging, specifically NP-complete. This means that no efficient algorithm is likely to exist for solving this problem in polynomial time. Then we restrict the preferences of individuals to be binary, meaning that each agent can have at most two objects in their preferences. The complexity result remains for the restricted case. To address these intractable cases, we propose a Mixed Integer Linear Programming (MILP) formulation, providing a practical approach for solving small to medium-sized instances of these problems. To the best of our knowledge, our MILP formulation is novel. More precisely, we provide a novel formalization of

[1] According to the English Housing Survey (2021–2022), there were approximately 210,000 housing moves annually, with over half involving transfers from one social rented property to another (Department for Levelling Up, Housing and Communities (2023)).

[2] An allocation is Pareto efficient if there is no other allocation where at least one agent is strictly better off and no agent is worse off. An allocation is individually rational if each agent receives an object that is at least as preferred as their original endowment.

[3] See Commissione Interministeriale per la Giustizia nel Sud e Isole (2022), which highlights significant turnover challenges in some regions.

[4] Maximum version of MINDIST is addressed by Salman (2025).

individual rationality and Pareto efficiency in terms of linear inequalities. It is important because these concepts are crucial for many object allocation problems and hence our formalization can be used to solve other related problems. Although MINDIST is computationally intractable, our MILP formulation is highly practical for many real-world small to medium-sized instances, offering solutions without imposing significant computational demands.

Related Literature Biró and Gudmundsson (2021) studies the problem of finding a Pareto efficient allocation that maximizes total welfare within Pareto efficient ones. They call this problem as ConstrainedWelfareMax and the outcome of this problem is a constrained welfare-maximizing allocation. The welfare value of agent i receiving object o is given by some number $\alpha(i, o)$. Total welfare of a given matching is then the sum of the individual welfare values over all agents.

They prove that the problem of finding a constrained welfare-maximizing allocation is NP-hard. It turns out that MINDIST is becoming a particular instance of their problem. Suppose we define a welfare matrix in which each cell represents the welfare derived by an individual from being assigned a particular object. For each agent i, objects less preferred than their endowment are assigned very small values, the endowment itself yields a welfare of 0, and any object preferred to the endowment yields a welfare of -1. In this setting, maximizing total welfare over the set of Pareto efficient and individually rational allocations becomes equivalent to minimizing the number of individuals who must be reallocated from their initial endowments. Hence, we prove a meaningful and more restricted case of the general problem still remains NP-complete.

Another closely related study is Abraham et al. (2004), which examines an object allocation problem where some agents may remain unassigned. They prove that finding a Pareto efficient allocation that minimizes the number of assigned agents is NP-complete.

A key distinction between their work and the current study lies in the problem focus. Abraham et al. (2004) shows that when all agents have initial endowments, all allocations have the same cardinality, and the Top Trading Cycle (TTC) algorithm, introduced by Shapley and Scarf (1974), yields a core allocation that is both individually rational and Pareto efficient. In contrast, MINDIST admits multiple individually rational and Pareto efficient allocations, which may differ in the number of agents who improve upon their initial endowments. As a result, the NP-completeness result in Abraham et al. (2004) does not directly apply to the complexity analysis in this study. To this end, this paper provides a Mixed Integer Linear Programming (MILP) formulation to solve small- and medium-sized instances of the NP-complete problem—an approach not addressed by Abraham et al. (2004).

Finally, this paper contributes to the literature on matching theory and market design, building upon foundational studies by Gale and Shapley (1962) and Shapley and Scarf (1974). For complexity results, see Manlove (2013). Surveys of matching theory and market design include Gusfield and Irving (1989), Roth and Sotomayor (1992), and Klaus et al. (2016), while practical applications are discussed in Biró (2017).

2 The Object Allocation Problem

Let I denote a finite set of agents and H a finite set of objects. Each object $h \in H$ has a positive integer capacity $q_h \in \mathbb{N}_+$, and we represent the collection of capacities as the profile $q = (q_h)_{h\in H}$. Each agent $i \in I$ has a complete and strict preference ranking P_i over the set of acceptable objects. We denote the full preference profile of all agents by $P= (P_i)_{i\in I}$. For each agent $i \in I$ and any two objects $h, h' \in H$, the notation $h\ P_i\ h'$ indicates that agent i strictly prefers object h over object h'. Additionally, we define the weak preference relation R_i, where $h\ R_i\ h'$ if and only if either $h\ P_i\ h'$ or $h = h'$. Finally, an object $h \in H$ is considered acceptable to an agent $i \in I$ if and only if $h\ R_i\ \omega(i)$, where $\omega(i)$ denotes agent i's initial endowment.

An allocation is a mapping $\mu : I \to H$ that assigns each agent $i \in I$ exactly one object $h \in H$. Every allocation must satisfy the capacity constraint:

$$|\{i \in I \mid \mu(i) = h\}| = q_h \quad \text{for each object } h \in H,$$

which states that the number of agents allocated to object h must equal its capacity q_h. We denote by ω the initial allocation (endowment), and we assume that every agent has an initial endowment, i.e., there is no agent $i \in I$ such that $\omega(i) = \emptyset$. For each object $h \in H$, let ω_h denote the set of agents initially endowed with h, defined as:

$$\omega_h = \{i \in I : \omega(i) = h\}.$$

We assume consistency in initial allocation and capacities, meaning that for each $h \in H$:

$$|\omega_h| = q_h.$$

Given these components, an object allocation instance is formally described by a tuple $\mathcal{I} = \langle I, H, q, P, \omega\rangle$.

An allocation μ is *individually rational (IR)* if no agent receives an object worse than their initial endowment.

Definition 1 (Individual Rationality (IR))**.** *Given an object allocation instance $\mathcal{I} = \langle I, H, q, P, \omega \rangle$, an allocation μ is individually rational if, for every agent $i \in I$,*

$$\mu(i) \; R_i \; \omega(i).$$

We denote by $M_{\mathcal{I}}^{IR}$ the set of all individually rational allocations for instance $\mathcal{I}$.

An allocation μ is *Pareto efficient (PE)* if there is no alternative allocation that makes at least one agent better off without making anyone worse off.

Definition 2 (Pareto Efficiency (PE))**.** *Given an object allocation instance $\mathcal{I}$, an allocation μ' Pareto dominates allocation μ if:*

- *for every agent $i \in I$, $\mu'(i) \; R_i \; \mu(i)$, and*
- *for at least one agent $i \in I$, $\mu'(i) \; P_i \; \mu(i)$.*

An allocation μ is Pareto efficient if no allocation Pareto dominates it.

We denote by $M_{\mathcal{I}}^{PE}$ the set of all Pareto efficient allocations for instance $\mathcal{I}$.

If an allocation is not Pareto efficient, there exists at least one subset of agents who can mutually benefit by exchanging their allocated objects.

Finally, given two allocations μ and μ', the distance between them is the number of agents who receive different objects:

Definition 3 (Distance between allocations)**.** *Given an object allocation instance and two allocations μ and μ', we define the distance between μ and μ' as:*

$$d(\mu, \mu') = |\{i \in I \mid \mu(i) \neq \mu'(i)\}|.$$

3 MINDIST

The aim of this study is to find an allocation μ for which the distance between μ and ω, i.e. $d(\mu, \omega)$, is minimum among all allocations that are PE and IR.

$$\min_{\mu \in M_{\mathcal{I}}^{PE} \cap M_{\mathcal{I}}^{IR}} d(\mu, \omega) \qquad \text{(MINDIST)}$$

We show that MINDIST turn out to be computationally intractable. To do this, first, we need to translate MINDIST to its equivalent decision problem.

Problem (DP-MINDIST)**.** *Given an object allocation instance* $\mathcal{I} = \langle I, H, q, P, \omega \rangle$ *and a number* $K \in \mathbb{N}$*, is there an allocation* $\mu \in M_{\mathcal{I}}^{IR} \cap M_{\mathcal{I}}^{PE}$ *such that* $d(\mu, \omega) \leq K$*?*

We have the following result.

Theorem 1. *The decision problem DP-MINDIST is NP-complete. This holds even if we restrict the capacity of each object to be equal to 1, i.e.* $q_h = 1$ *for all* $h \in H$*.*

Proof. See Appendix A.1. □

It is important to highlight that an object allocation instance where $q_h = 1$ for every object $h \in H$ represents a specific instance of the broader problem. Therefore, if addressing the complexity of DP-MINDIST proves to be challenging for such specialized cases, it becomes even more difficult in a more general setting.

Let us provide a brief outline of the proof of Theorem 1. The proof relies on a polynomial reduction from the NP-hard *vertex cover* problem. An instance of the vertex cover problem consists of an undirected graph $G = (V, E)$ and a positive integer k. Here V is a finite set of vertices and E contains a collection of edges, where every edge consists of two distinct vertices from V. A vertex cover V' of G is a subset of vertices such that every edge in E contains at least one vertex in V'.

Problem (Minimum Vertex Cover)**.** *Given a graph* $G = (V, E)$ *and an integer* k*, does there exist a vertex cover* V' *of* G *such that* $|V'| \leq k$*?* [5]

For the reduction, we start with an arbitrary instance $\langle G = (V, E), k \rangle$ of the minimum vertex cover problem and from this, we construct an instance of the DP-MINDIST problem. Next, we show that there exists a vertex cover V' with $|V'| \leq k$ if and only if the DP-MINDIST instance has a IR and PE allocation with distance less or equal to K.

Given the graph $G = (V, E)$, we first create for every vertex $v \in V$ three individuals, each with their endowments and preferences. Each vertex can be visualized as a small instance of an object allocation problem. The construction is such that at any PE and IR allocation either all three individuals will change their endowments (situation A) or two of these three individuals change their initial endowments (situation B).

[5] A reasonable lower bound for k is the cardinality of the maximal matching on the graph. See Garey and Johnson (1979), page 134.

If there is a vertex cover of size k, then we assign all individuals that are related to these vertices to situation A, while all individuals related to vertices not in the vertex cover will be related to situation B. In this way, we are able to construct a PE and IR allocation of distance less than or equal to $3k + 2(|V| - k) = k + 2|V|$. Choosing $K = k + 2|V|$ for the DP-MINDIST instance shows that any "yes" instance of the minimum vertex cover problem gives a "yes" instance for the DP-MINDIST problem. The proof is finalized by showing that the reverse also holds: a PE and IR allocation in which the distance is less than or equal to K gives rise to a vertex cover for which the size is less than or equal to k.

4 Binary preferences

As a natural attempt after showing that MINDIST is NP-complete, we try to restrict the problem to see if NP-hardness holds even for the restricted version.

In this section, we look at imposing a restriction on the preferences. In particular, we consider the situation where preferences are restricted to be *binary*. In a binary preference profile, each agent is restricted to rank at most one object above their endowment.

Definition 4. *A preference profile* $P = (P_1, ..., P_{|I|})$ *is called binary if for each* $i \in I$, $|\{h \in H : h \; R_i \; \omega(i)\}| \leq 2$. *An object allocation instance* $\mathcal{I} = \langle I, H, q, P, \omega \rangle$ *is called an object allocation instance with binary preferences if the preference profile* P *is binary.*

Problem (DP-MINDIST-BP)**.** *Given an object allocation instance with binary preferences* $\mathcal{I} = \langle I, H, q, P, \omega \rangle$ *and a number* K, *does there exist a Pareto efficient and individually rational allocation* $\mu \in M_{\mathcal{I}}^{IR} \cap M_{\mathcal{I}}^{PE}$ *such that* $d(\mu, \omega) \leq K$*?*

We have the following result.

Theorem 2. *The decision problem DP-MINDIST-BP is NP-complete.*

Proof. See Appendix A.2. □

The approach in the proof of Theorem 2 is similar to the proof of Theorem 1, but the construction is quite a bit more intricate. Specifically, we again use a reduction from the vertex cover problem. Same as in the proof of Theorem 1, we create for every vertex several individuals with endowments and preferences.

As with the proof of Theorem 1, a connection is drawn between two key parameters: K, representing the upper bound on the distance between the endowment and the PE and IR allocation, and the upper bound k for the size of the vertex cover.

It is also worth noting that the construction in the proof now involves instances that require objects whose capacities are larger than 1. The reason for this is quite simple. In particular, if for an object allocation instance with binary preferences, each object has a unique capacity, then there is a unique PE and IR allocation (that can be found by the TTC algorithm).

5 Integer programming formulation

We have shown that 1 and 2 are computationally intractable. More precisely, the fastest algorithm one can design to solve these problems can still suffer of running time for the worst case scenarios. However, there might be some instances of these problems such that a non-polynomial algorithm can provide the solution fast. Therefore, a follow up question is whether one can find a non-polynomial algorithm that provides a solution for instances that are not too large. In this section, we answer this question by providing a Mixed Integer Linear Programming (MILP) formulations of the complex problems. A MILP is a linear programming problem where some of the variables are restricted to be integer valued (for our case either 0 or 1). Despite the fact that MILP formulations for NP-hard problems are generally hard to solve, there exists pertinent software packages that can compute solutions for moderately sized instances in a reasonable amount of time.[6]

To set up the MILP formulation, we utilize the following notation. For every agent $i \in I$ and every object $h \in H$, we construct a binary variable $x(i,h) \in \{0,1\}$. The intuition is that $x(i,h) = 1$ if and only if h is allocated to the agent i. The distance between an allocation (now determined by x) and the endowment is then given by $|I| - \sum_i x(i, \omega(i))$. So, maximizing $\sum_i x(i, \omega(i))$ amounts to minimizing the distance between the allocation given by x and the endowment.

As x is an allocation, it should satisfy the restrictions that for all i, $\sum_h x(i,h) = 1$ and that for all h, $\sum_i x(i,h) = q_h$ which means that every individual is assigned to an object and the total number of individuals assigned to each object equals the capacity of the object. These are given as restrictions (IP-1) and (IP-2).

Next, we formalize the IR and PE restrictions in terms of linear inequalities. To the best of our knowledge, our formalizations of IR and PE in terms of linear

[6]See, for example, the GUROBI optimization package (https://www.gurobi.com/). Last access to the website is on the 3rd of June, 2025.

inequalities are novel.[7] To do this, we construct, for each $i \in I$, the variable $u(i) \in [0,1]$ which capture the preferences of the individuals. For a given allocation, $u(i) > u(j)$ whenever agent i prefers to have the object of agent j over their current object in the allocation. In the proof of Theorem 3, we show how these numbers can be constructed efficiently.

We also encode preferences in the following way. For every $i \in I$, and $h, k \in H$ with $h \neq k$, we define $pr_i(h,k) = 1$ if and only if i prefers h over k, i.e. $h \ P_i \ k$. We also define $r_i(h, \omega(i)) = 1$ if and only if i prefers h over $\omega(i)$ or $h = \omega(i)$, i.e. $h \ R_i \ \omega(i)$.

The IR condition can simply be stated as $\sum_h x(i,h) r_i(h, \omega(i)) = 1$, which means that i should receive an object that is at least as good as their endowment. This is given by restriction (IP-3) below. Note that this restriction is stronger than (IP-2), so we could also omit the latter one.

Constraint (IP-4) guarantees that the allocation satisfies Pareto efficiency. In particular, the constraint guarantees that there will be no cycle on the envy graph of the object allocation instance.[8] The idea behind the constraint is that if i points to j in the envy graph (i.e. if $x(i,h) + x(j,k) + pr_i(k,h) = 3$), then $u(i) > u(j)$. As such, any cycle $i \to j \to k \ldots \to i$ in the envy graph would imply that $u(i) > u(j) > \ldots > u(i)$, which is impossible. Therefore, if (IP-4) is satisfied, then the allocation satisfies Pareto efficiency.

The solution of the following problem then solves the MINDIST problem. Here M is a very big number and ϵ is a very small number.

$$\max_{x(i,h), u(i) \ \forall i \in I, h \in H} \sum_{i \in I} x(i, \omega(i)) \tag{MINDIST-ILP}$$

$$\text{s.t.} \quad \sum_{i \in I} x(i,h) = q_h \qquad \forall h \in H \tag{IP-1}$$

$$\sum_{h \in H} x(i,h) = 1 \qquad \forall i \in I \tag{IP-2}$$

$$\sum_{h \in H} x(i,h) r_i(h, \omega(i)) = 1 \qquad \forall i \in I \tag{IP-3}$$

$$M\left[x(i,h) + x(j,k) + pr_i(k,h) - 3\right] \leq u(i) - u(j) - \epsilon \quad \forall i,j \in I, \forall h,k \in H \tag{IP-4}$$

[7] Biró and Gudmundsson (2021) also formulates PE in terms of linear inequalities. However, our methodology does not rely on theirs.

[8] An envy graph of the allocation μ is a directed graph $G_\mu = (V_\mu, E_\mu)$ where the set of vertices V_μ is composed of the agents and from each $v \in V_\mu$ there is an edge to $v' \in V_\mu$ if and only if $\mu(v') \ P_v \ \mu(v)$.

Theorem 3. *Given an object allocation instance $\mathcal{I} = \langle I, H, q, P, \omega \rangle$,*

- *if $(x(i,h), u(i))_{i \in I, h \in H}$ is a solution to MINDIST-ILP, then the matching μ, where $\mu(i) = h$ if and only if $x(i,h) = 1$, solves MINDIST.*
- *If μ solves MINDIST, then there is a solution $(x(i,h), u(i))_{i \in I, h \in H}$ to MINDIST-ILP such that $x(i,h) = 1$ if and only if $\mu(i) = h$.*

6 Conclusion

This paper investigated the problem of minimizing the distance between the initial endowments and the final PE and IR allocations. We showed that MINDIST is NP-complete. This is true even under the assumption that each object has a unique capacity. Also, it turns out that MINDIST remains NP-hard if preferences are restricted to be binary. Finally, we introduced a novel linear integer programming to solve MINDIST for relatively small instances.

A first extension of this research could involve investigating MINDIST under additional restrictions to identify tractable cases. Secondly, it would be beneficial to conduct a real-life application. Finally, extending MINDIST framework to a two-sided matching market and analyzing it using the corresponding equilibrium concept—namely, stability—represents another promising avenue for future research.

A Proofs of results in main text

A.1 Proof of Theorem 1

Proof. Let us first show that it is in NP. For an instance $\mathcal{I} = \langle I, H, \omega, P, q \rangle$ and an allocation μ, one can easily show that one can verify in polynomial time that $\mu \in M_{\mathcal{I}}^{IR} \cap M_{\mathcal{I}}^{PE}$ and $d(\omega, \mu) \leq K$.

For the NP-hardness part, we use a reduction from the minimum vertex cover problem. We denote a graph by $G = (V, E)$ where V is the set of vertices and E is the set of edges, which is a set of elements (i,j) with $i, j \in V$ (and $i \neq j$). Here we look at undirected edges, so the edge (i,j) is the same as the edge (j,i).

Problem (Minimum Vertex Cover)**.** *Given an undirected graph $G = (V, E)$, does there exist a subset $V' \subset V$ of no more than k vertices such that for every edge $(i,j) \in E$ either $i \in V'$ or $j \in V'$.*

The minimum vertex cover problem is known to be NP-hard.[9] Now, let $G = (V, E)$ be an instance of the minimum vertex cover problem. We need to construct an instance $\mathcal{I} = \langle I, H, \omega, P, q\rangle$ of MINDIST, such that the minimum vertex cover instance is satisfiable if and only if the instance of MINDIST is satisfiable.

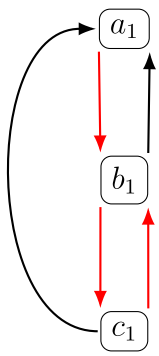
Assume that there are N vertices $v_1, \ldots, v_N$. For every vertex $v_i \in V$ we consider 3 individuals a_i, b_i, c_i endowed with objects A_i, B_i and C_i. You can visualize each vertex v_i as a small object allocation instance that is composed of 3 agents a_i, b_i and c_i. It is mostly called a gadget in NP-complete proofs.

Preferences are the following

individual	endowment	preferences
a_i	A_i	$B_i, \{A_j : (v_i, v_j) \in E\}, A_i$
b_i	B_i	C_i, A_i, B_i
c_i	C_i	B_i, A_i, C_i

To explain the notation used for the preferences of the individuals, consider individual a_i. Their top choice is B_i, then follows all objects $\{A_j : (i, j) \in E\}$ in some arbitrary order. Next is their endowment A_i. Finally all other objects (omitted from the notation) are ranked below A_i. As we only look for individual rational allocations, there is no need to define preferences below the individuals' own endowment, so we omit them from the notation. As a second example, for agent b_i, their top choice is C_i, next A_i and then their endowment B_i. All other options are ranked below B_i.

The envy graph among the nodes $\{a_1, b_1, c_1\}$ is given below. Here an individual points to another individual if she prefers this person's allocation (endowment) over their own. Arrows in red point to the most preferred option.



The figure indicates that there are 2 possible cases. One possible allocation is that b_i receives C_i and c_i receives B_i. Once this exchange is established neither b_i nor

[9] It is problem GT1 in Garey and Johnson (1979).

c_i is willing to make another exchange (as they get their most preferred object). This leaves a_i with A_i, which she might or might not exchange with an individual associated with another vertex.

A second possible allocation is that b_i receives C_i, c_i receives A_i and a_i receives B_i. If so, neither a_i nor b_i is still willing to make another exchange (as they get their best choice). Individual c_i would prefer B_i, but this object is currently taken by a_i, who will not be willing to exchange it for something else.

To finalize the instance, we set $K = 2N + k$.

In total we have an instance $\mathcal{I} = \langle I, H, \omega, P, q\rangle$ with $3N$ individuals and objects, the capacity of each object in $\{A_1, ..., A_N, B_1, ..., B_N, C_1, ..., C_N\}$ is 1. If $G = (V, E)$ grows, the size of the instance will grow linearly. Hence the size of the instance is polynomial in the size of the minimum vertex cover instance.

We need to show that there is a vertex cover of no more than k elements if and only if there is an allocation μ in $M^{PE}_{\mathcal{I}} \cap M^{IR}_{\mathcal{I}}$ for which $d(\omega, \mu) \leq K = 2N + k$.

For the first part, assume that V' is a vertex cover with $|V'| \leq k$. We define an allocation μ in the following way, depending on whether $v_i \in V'$ or not.

individual	endowment	μ	condition
a_i	A_i	B_i	$v_i \in V'$
b_i	B_i	C_i	$v_i \in V'$
c_i	C_i	A_i	$v_i \in V'$
a_i	A_i	A_i	$v_i \notin V'$
b_i	B_i	C_i	$v_i \notin V'$
c_i	C_i	B_i	$v_i \notin V'$

For each vertex in the vertex cover, V', there will be 3 individuals who will change their initial allocations. For each vertex outside of the vertex cover, there will be 2 individuals who will change their initial endowments. Hence, the total number of changes will be $2(N - |V'|) + 3|V'| = 2N + |V'| \leq 2N + k$.

We need to show that μ is both individually rational and Pareto efficient.

Suppose for a contradiction that there is a coalition S and each agent in this coalition can be better off by exchanging their current objects (their objects in μ) or initial endowments among each other. Consequently, these exchanges lead to the creation of a new allocation, denoted as μ', which is preferred to the existing allocation μ by the members of S.

- If $c_i \in S$ and $v_i \in V'$ then $\mu'(c_i) = B_i$ but then a_i (who has B_i in μ) or b_i

(who has B_i as their initial endowment) is also in S. This is impossible as both individuals gets their best choice in μ.

- If $c_i \in S$ and $v_i \notin V'$, this is impossible as B_i is c_i's best choice.
- If $b_i \in S$, this is impossible for the same reason.
- If $a_i \in S$ and $v_i \in V'$, this is once more impossible, again for the same reason.
- Finally, assume $a_i \in S$ and $v_i \notin V'$. Then either $\mu'(a_i) = B_i$ or $\mu'(a_i) = A_j$ for some $(i, j) \in E$. If $\mu'(a_i) = B_i$, then c_i or b_i is also in S. But this is impossible because c_i or b_i already has their best choice. So, it must be that $\mu'(a_i) = A_j$ for some $(i, j) \in E$. In μ, the object A_j is either with c_j or a_j has it as their initial endowment. If $(v_i, v_j) \in E$ then a_j gets their best object because he is outside of V'. If it is with c_j, then it must be that $\mu'(c_j) = B_j$ and the object B_j is with a_j. Then $a_j \in S$, which is impossible as a_j receives their best choice B_j. If a_i is in S with a_j, then $(v_i, v_j) \in E$. $\mu(a_j) = A_j$ means that $v_j \notin V'$. If $(v_i, v_j) \in E$, then both v_i and v_j should be in V' and this contradicts that with $v_i \notin V'$.

This shows that $\mu \in M_{\mathcal{I}}^{PE} \cap M_{\mathcal{I}}^{IR}$.

For the reverse, assume that there is an allocation $\mu \in M_{\mathcal{I}}^{IR} \cap M_{\mathcal{I}}^{PE}$ such that $d(\omega, \mu) \leq 2N + k$. We need to show that there is a vertex cover of size less than or equal to k.

First note that the number of individuals among $\{a_i, b_i, c_i\}$ that do not receive their initial object in μ must be at least 2: in particular it must include both b_i and c_i. If not then either $\mu(b_i) = B_i$ and $\mu(c_i) \in \{C_i, A_i\}$ in which b_i and c_i would exchange their initial objects and both of them would be better off, or $\mu(c_i) = C_i$ and $\mu(b_i) \in \{B_i, A_i\}$ in which, again b_i and c_i would exchange their initial objects and both of them would be better off.

Therefore, $d(\omega, \mu) \geq 2N$. Define the set $V' \subset V$ such that $v_i \in V'$ if and only if all individuals in $\{a_i, b_i, c_i\}$ receive an object in μ distinct from their initial endowments. Notice that

$$d(\omega, \mu) = 3|V'| + 2(N - |V'|) = 2N + |V'| \leq 2N + k.$$

Hence, $|V'| \leq k$.

Finally, we need to show that V' is a vertex cover. Let $v_i, v_j \notin V'$ and assume, towards a contradiction, that $(v_i, v_j) \in E$. Then as both $\{a_i, b_i, c_i\}$ and $\{a_j, b_j, c_j\}$

only have two individuals that do not receive their initial allocation, it must be that $\mu(a_i) = A_i$ and $\mu(a_j) = A_j$. But then a_i and a_j would exchange their initial objects (or their current objects) and both of them would be better off. This contradicts with the assumption that μ was PE. □

A.2 Proof of Theorem 2

Proof. The proof also uses a reduction from the NP-hard vertex cover problem.

Vertex cover: *Given a network (V, E) and a number k, does there exist a subset $V' \subseteq V$ of vertices of size $|V'| \leq k$ such that for all edges $(i, j) \in E$ either $i \in V'$ or $j \in V'$.*

Consider an instance $\langle (V, E), k \rangle$ of the vertex cover problem. Without loss of generality, we can assume that all vertices have at least one edge. For any v, let k_v be the degree of v (i.e. number of edges adjacent to v). Let $w_1, \ldots, w_{k_v}$ be an enumeration of the neighbours of v (in no particular order).

We construct an instance $\{\langle I, H, q, P, \omega \rangle, W\}$ of the DP-MINDIST-DP problem.

Agent $i \in I$ has an endowment, denoted $\omega(i) \in H$ and a single object, denoted $p(i) \in H$, that they prefer to their current endowment.

- For every $v \in V$, we create the following collection of individuals.
  - we create two individuals $a_{v,1}$ and $a_{v,2}$.
  - For every neighbour w_i of v ($i = 1, \ldots, k_v$), create 4 individuals b_{v,w_i}, c_{v,w_i}, d_{v,w_i} and e_{v,w_i}.
  - We create $4k_v$ additional individuals $f_{v,1}, \ldots, f_{v,4k_v}$. Remember that k_v is the degree of vertex v.
  - Let $Z = \max\{k, \max_{v \in V} 4k_v\}$. For every neighbour w_i of v ($i \leq k_v$), we create Z individuals, $g_{v,w_i,1}, \ldots, g_{v,w_i,Z}$.
- The endowments and preferences of the various individuals are given in the table below.

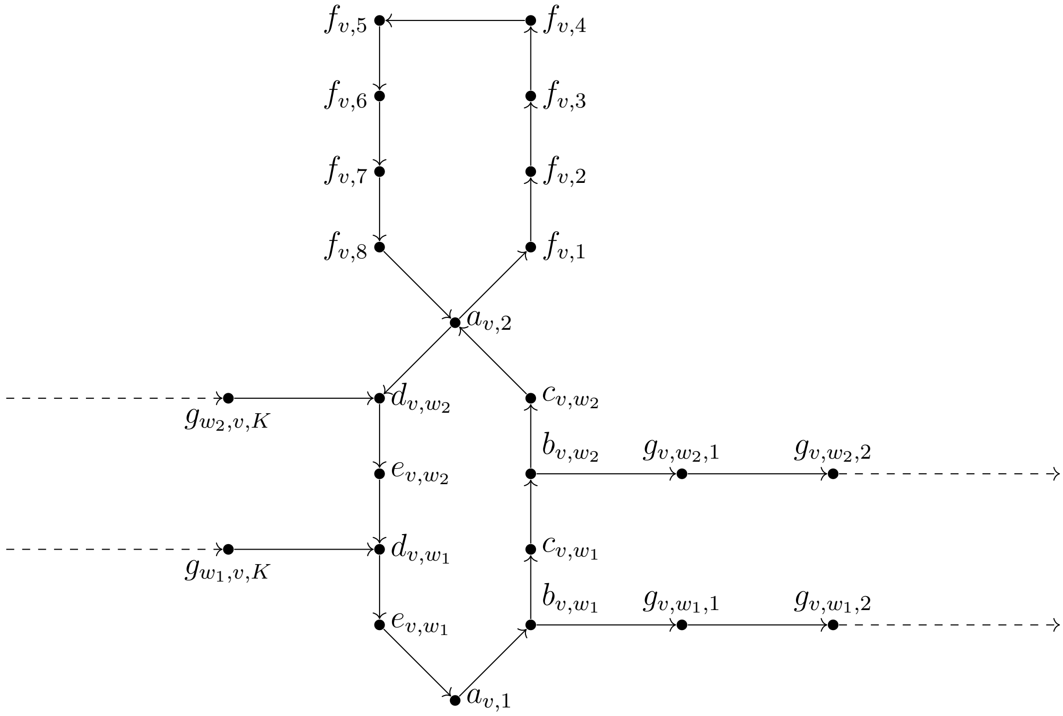
individual i	endowment $\omega(i)$	preference $p(i)$
$a_{v,1}$	$A_{v,1}$	B_{v,w_1}
b_{v,w_i} $(i = 1, \ldots, k_v)$	B_{v,w_i}	C_{v,w_i}
c_{v,w_i} $(i = 1, \ldots, k_v - 1)$	C_{v,w_i}	$B_{v,w_{i+1}}$
$c_{v,w_{k_v}}$	$C_{v,w_{k_v}}$	$A_{v,2}$
$a_{v,2}$	$A_{v,2}$	$D_{v,w_{k_v}}$
d_{v,w_i} $(i = 1, \ldots, k_v)$	D_{v,w_i}	E_{v,w_i}
e_{v,w_i} $(i = 1, \ldots, k_v - 1)$	E_{v,w_i}	$D_{v,w_{i-1}}$
e_{v,w_1}	E_{v,w_1}	$A_{v,1}$
$f_{v,1}$	D_{v,w_1}	$F_{v,1}$
$f_{v,i}$ $(i = 2, 4k_v - 1)$	$F_{v,i}$	$F_{v,i+1}$
$f_{v,4k_v}$	$F_{v,4k_v}$	$A_{v,2}$
$g_{v,w_i,1}$ $(i = 1, \ldots, k_v)$	C_{v,w_i}	$G_{v,w_i,2}$
$g_{v,w_i,j}$ $(i = 1, \ldots, k_v, j = 1, \ldots Z - 1)$	$G_{v,w_i,j}$	$G_{v,w_i,j+1}$
$g_{v,w_i,Z}$ $(i = 1, \ldots, k_v)$	$G_{v,w_i,Z}$	$D_{w_i,v}$

- We set $K = 2Z|E| + \sum_{v \in V} 4k_v + |V| - k$

This instance is polynomial in the size of the vertex cover instance.

For ease of notation we call $\{b_{v,w_i}, i = 1, \ldots, k_v\}$ the b_v individuals. Similarly, we call $\{c_{v,w_i}, i = 1, \ldots, k_v\}$ the c_v individuals, $\{d_{v,w_i}, i = 1, \ldots, k_v\}$ the d_v individuals, $\{e_{v,w_i}, i = 1, \ldots, v_k\}$ the e_v individuals, $\{f_{v,i}, i = 1, \ldots, 4k_v\}$ the f_v indivivduals, $\{g_{v,w_i,\ell}, i = 1, \ldots, k_v, \ell = 1, \ldots, Z\}$ the g_v individuals, and $\{a_{v,1}, a_{v,2}\}$ the a_v individuals .

The figure below gives an illustration of the the graph of the instance induces by a node v of degree 2, that has neighbours w_1 and w_2.



The idea behind the proof is the following. As the picture hints at, there are three types of cycles. A first type involves $a_{v,2}$ and the f_v individuals. This cycle is of length $4k_v + 1$. A second type of cycle involves the a_v, b_v, c_v, d_v and e_v individuals. This one is slightly longer and of length $4k_v + 2$. Finally, if there is an edge (v, w) there is a cycle that connects $a_{v,1}$ to $b_{v,w}$, via g_v individuals to $d_{w,v}$ then via $a_{w,1}$ to $b_{w,v}$ and via the g_w indivdiuals to $d_{v,w}$ and then back to $a_{v,1}$. These long cycles have length at least $2Z$.

To establish the proof, we need to show that $\langle (V, E), k\rangle$ is a yes instance of the vertex covering decision problem if and only if $\{\langle I, H, q, P, \omega\rangle, K\}$ is a yes instance of DP-MINDIST-DP.

($\Rightarrow$) Let $\langle (V, E), k\rangle$ be a yes instance of vertex cover and let $V' \subseteq V$ be a vertex cover with $|V'| \leq k$. Consider the allocation μ where for all $v \in V'$, $\mu(i) \neq \omega(i)$ if and only if i is an a_v, b_v, c_v, d_v or e_v individual and for $v \notin V'$, $\mu(i) \neq \omega(i)$ if and only if i is either $a_{v,2}$ or an f_v individual. In other words, we implement the second type of cycle if $v \in V'$ and the first type if $v \notin V'$. Notice that this is indeed a valid allocation.

The total number of individuals equals $2Z|E| + \sum_{v\in V}(8k_v + 2)$. As such, the total number of individuals that keep their endowment is given by:

$$
\begin{aligned}
&2Z|E| + \sum_{v\in V}(8k_v+2) - \sum_{v\in S}(4k_v+2) - \sum_{v\notin V'}(4k_v+1),\\
&= 2Z|E| + \sum_{v\in V}(4k_v+1) - |V'|,\\
&\geq 2Z|E| + \sum_{v\in V} 4k_v + |V| - k = K.
\end{aligned}
$$

It is obvious that μ is individually rational.

We still need to show that the allocation is Pareto efficient. If not, there must be a cycle in the envy grapy after removing all individuals that do not receive their endowment. We work towards a contradiction:

- notice that $a_{v,2}$ is never allocated their endowment, so can not be part of a remaining cycle.
- if there is an f_v node that is part of a remaining cycle, then all f_v nodes must be part of the cycle. But this implies that $a_{v,2}$ must also be part of the cycle, a contradiction.
- if $c_{v,w_{k_v}}$ is allocated their endowment and is part of a cycle, then $a_{v,2}$ must also be part of a cycle, a contradiction.
- We now show that $b_{v,w_i}, d_{v,w_i}, e_{v,w_i}$ and c_{v,w_i} are not part of the cycle for all $i = 1,\dots,k_v$ by backward induction on i. We know it is true for $c_{v,w_{k_v}}$.
  - To show it is true for $b_{v,w_{k_v}}$. Assume that $b_{v,w_{k_v}}$ is allocated their endowment and is part of a remaining cycle, then $v \notin V'$. Also, we have that either $c_{v,w_{k_v}}$ is part of the cycle, or $g_{v,w_{k_v},1}$ is. The first is impossible, by induction, so the second must be the case. But then $g_{v,w_{k_v},2},\dots,g_{v,w_{k_v},Z}$ and $d_{w_{k_v},v}$ are also part of the cycle. The latter implies that $w_{k_v} \notin V'$, which contradicts the assumption that V' is a vertex cover.
  - To show that it is true for $d_{v,w_{k_v}}$. Assume $d_{v,w_{k_v}}$ is part of a cycle, then either $a_{v,2}$ is or $g_{w_{k_v},v,K}$ is. The first is impossible so $g_{w_{k_v},v,K}$ must receive their endowment. This means that $g_{w_{k_v},v,K-1},\dots,g_{w_{k_v},v,1}$ and $b_{v,w_{k_v}}$ must also be part of the cycle. The latter implies that $w_{k_v} \notin V'$, contradicting the assumption that V' is a vertex cover.
  - For the induction step. If c_{v,w_i} is part of the cycle then $b_{v,w_{i+1}}$ must also be, which contradicts the induction hypothesis. If e_{v,w_i} is part of the

cycle, then $v \notin V'$ and d_{v,w_i} is also. Then either $e_{v,w_{i-1}}$ is part of the cycle or $g_{w_i,v,Z}$ is. The first is impossible by the induction hypothesis. So $g_{w_i,v,Z-1}, \ldots, g_{w_i,v,1}$ and $b_{w_i,v}$ are also part of the cycle. But then $w_i \notin V'$, contradicting the assumption that V' is a vertex cover. Finally, if b_{v,w_i} is part of a cycle then $v \notin V'$ and either $c_{v,w_{i+1}}$ or $g_{v,w_i,1}$ is part of the cycle. The first is impossible by induction, while the second leads to d_{v,w_i} being part of the cycle and hence $w_i \notin V'$, a contradiction.

- if $a_{v,1}$ is part of the cycle, then b_{v,w_1} is also, which is a contradiction.
- if $g_{v,w_i,j}$ for some $i = 1, \ldots, k_v$ and $j \leq Z$ is part of the cycle then $g_{v,w_i,1}, \ldots, g_{v,w_i,Z}$ are also and, hence, also b_{v,w_i} and $d_{w_i,v}$. This, however, implies that both $v, w_i \notin V'$, a contradiction with the assumption that V' is a vertex cover.

($\Leftarrow$) Now, for the reverse, assume that $\{\langle I, H, q, P, \omega\rangle, K\}$ is a yes instance. Let μ be an allocation that solves the instance.

We first establish some useful results.

Lemma 1. *Let μ be a solution to the instance $\{\langle I, H, q, P, \omega\rangle, K\}$ then among the $a_v, b_v, c_v, d_v, e_v, f_v$ individuals, there are at least $4k_v + 1$ individuals that do not keep their initial endowment.*

Proof. If $a_{v,2}$ keeps their initial endowment, then so do all f_v individuals. But this implies that we have a cycle among the f_v individuals and $a_{v,2}$, which contradicts Pareto efficiency. As such, $a_{v,2}$ does not keep their initial endowment. Then either she gets the endowment of $f_{v,1}$ or from $d_{v,w_{k_v}}$ and they give their endowment to either $f_{v,v_{k_v}}$ or $c_{v,w_{k_v}}$.

If she gets the endowment of $f_{v,1}$ then $f_{v,1}$, in turn, gets the endowment of $f_{v,2}$ and so on. This implies that all f_v individuals do not keep their initial endowment, so the number of individuals that do not have their initial endowment is at least $4k_v + 1$ what we needed to show.

If $a_{v,2}$ gives their endowment to $f_{v,4k_v}$ then $f_{v,4k_v}$ gives their endowment to $f_{v,4k_v-1}$ and so on. This implies again that no f_v individual keeps their endowment so the number of individuals that do not get their initial endowment is again at least $4k_v + 1$ what we needed to show.

If $a_{v,2}$ gives their endowment to $c_{v,w_{k_v}}$ and gets the endowment of $d_{v,w_{k_v}}$, then d_{v,w_k} gets the endowment of $e_{v,w_{k_v}}$, who in turn gets the endowment of $d_{v,w_{k-1}}$ and so on until $a_{v,1}$. So none of the a_v, c_v, d_v keep their endowment.

Also as $a_{v,2}$ gives their endowment to $c_{v,w_{k_v}}$, she, in turn must give their endowment to $b_{v,w_{k_v}}$, who in turn must give their endowment to $c_{v,w_{k_v-1}}$ and so on until $a_{v,1}$. So also none of the b_v, c_v individuals keep their endowment. As such at least $4k_v + 2$ individuals do not keep their endowment. □

Lemma 2. *Let μ be a yes solution to $\{\langle I, H, q, P, \omega\rangle, K\}$. If $a_{v,1}$ does not keep their initial endowment, then among the a_v, b_v, c_v, d_v, e_v and g_v individuals, there are at least $4k_v + 2$ individuals that do not keep their initial endowment.*

Proof. Assume that $a_{v,1}$ does not keep their initial endowment. Then she must have received the endowment of b_{v,w_1}. This individual, must have reveived the endowment of either c_{v,w_1} or from $g_{v,w_1,1}$.

In the latter case, $g_{v,w_1,2}, \ldots, g_{v,w_1,Z}$ also do not keep their endowment, which implies that at least $4k_v + 2$ individuals do not keep their initial endowment.

In the former case, c_{v,w_1} receives their endowment from b_{v,w_2}. Then b_{v,w_2} gets their endowment from c_{v,w_3} or from $g_{v,w_2,1}$. The latter implies that at least $4k_v + 2$ individuals do not keep their initial endowment. We can continue until we arrive at $c_{v,w_{k_v}}$.

Now, $c_{v,w_{k_v}}$ must receive their endowment from $a_{v,2}$, who in turn must have received their endowment from either $f_{v,4k_v}$ or from $d_{v,w_{k_v}}$. The former implies that all f_v's do not receive their endowment, therefore passing the threshold of $4k_v + 2$. The second implies that $d_{v,w_{k_v}}$ received the endowment of $e_{v,w_{k_v}}$. This person received the endowment of $d_{v,w_{k_v-1}}$ and so on, until $a_{v,1}$. Conclude that in total, at least $4k_v + 2$ individuals did not receive their initial endowment. □

Lemma 3. *Let μ be a yes solution to $\{\langle I, H, q, P, \omega\rangle, K\}$. Then among the $a_v, b_v, c_v, e_v, d_v, f_v$ and g_v individuals, there are no more than $4k_v + 2$ individuals that do not keep their endowments.*

Proof. First let us show that if more than $4k_v + 2$ individuals do not keep their endowment, then there are in fact more than $4k_v + 1 + Z$ individuals that do not keep their initial endowment.

If $a_{v,2}$ gets the endowment of $f_{v,4k_v}$ or gives their endowment to $f_{v,4k_v}$ then then all f_v individuals do not keep their endowment. As this only adds up to $4k_v + 1$ individuals, there must be at least one other individual that does not keep their initial endowment.

- If $a_{v,1}$ does not keep their endowment, then neither does b_{v,w_1}. This means that b_{v,w_1} obtained their endowment from $g_{v,w_1,1}$ or from c_{v,w_1}. If the first is the case then $g_{v,w_1}, \ldots, g_{v,w_1,Z}$ also do not keep their endowment so in total more than $4k_v + 1 + Z$ individuals changed their endowment. So towards a contradiction, assume the latter is the case. We can continue in a similar way until in the end, we must have that c_{v,w_k} does not keep their endowment, but then, they must have received their endowment from $a_{v,2}$, which is impossible.

- If any of the b_v or c_v individuals did not keep their endowment, we arrive at a similar outcome.
- If any of the d_v or e_v individuals did not keep their endowment, we can trace the cycle and conclude that $a_{v,1}$ also did not keep their endowment, so we are back at the first case.

If $a_{v,2}$ gets the endowment of $d_{v,w_{k_v}}$ and gives their endowment to c_{v,w_k}, then d_{v,w_k} gets the endowment of e_{v,w_k}, who gets the endowment from $d_{v,w_{k-1}}$ and so on until a_{v_1}. Also c_{v,w_k} gives their endowment to b_{v,w_k} who gives their endowment to $c_{v,w_{v_k-1}}$ and so on until $a_{v,1}$. This amounts to exactly $4k_v + 2$ individuals. Also all individuals f_v must now keep their endowment. As such, to obtain more than $4k_v + 2$ individuals that do not keep their endowment, it must be that there is a $g_{v,w_i,\ell}$ that does not keep their endowment. But then also $g_{v,w_i,1}, \ldots, g_{v,w_i,Z}$ do not keep their endowment. So the total number of people not keeping their endowment is more than $4k_v + 1 + Z$.

To finish the proof. Assume by contradiction that there is a $v' \in V$ such that at least $4k_{v'} + 2$ individuals do not keep their endowment. The previous reasoning then shows that the total number of people that do not keep their endowment must be more than

$$\sum_{v \in V}(4k_v + 1) + Z \geq \sum_{v \in V}(4k_v + 1) + k.$$

As such, the number of people that do keep their endowment is less than

$$2Z|E| + \sum_{v \in V}(8k_v + 2) - \sum_{v \in V}(4k_v + 1) - k = 2Z|E| + \sum_{v} 4k_v + |V| - k = K,$$

a contradiction. $\square$

To continue the proof consider the set $V' \subseteq V$ such that $v \in V'$ if and only if $a_{v,1}$ does not keep their initial endowment.

Let us first show that $|V'| \leq k$. From the lemmata above, we have that the total number of individuals that do not keep their endowments is greater than or equal to:

$$\sum_{v \in V} (4k_v + 1) + |V'|.$$

So the total number of individuals that keep their endowment is less than or equal to:

$$2Z|E| + \sum_{v \in V} (8k_v + 2) - \sum_{v \in V} (4k_v + 1) - |V'| = 2Z|E| + \sum_{v \in V} 4k_v + |V| - |V'| = K + k - |V'|.$$

This number is also greater or equal to K, so:

$$K \leq K + k - |V'| \Leftrightarrow |V'| \leq k.$$

Lemma 4. *For $v \in V$, if $a_{v,1}$ keeps their endowment (i.e. $v \notin V'$) then among the $a_v, b_v, c_v, d_v, e_v, f_v$ and g_v individuals exactly the individuals $a_{v,2}$ and the individuals in f_v change their endowment.*

Proof. From the previous lemmata, we have shown that for every v, the number of individuals among the a_v, b_v, c_v, d_v, f_v and g_v individuals that do not keep their endowment is either $4k_v + 1$ or $4k_v + 2$.

Assume $a_{v,1}$ keeps their endowment ($v \notin V'$). Then so does e_{v,w_1}, d_{v,w_1} and so on until $d_{v,w_{v_k}}$. As $a_{v,2}$ does not keep their endowment, it must be that she gets the endowment of $f_{v,1}$. As such, none of the f_v individuals keep their endowment. The individual $a_{v,2}$ together with the f_v individuals give $4k_v + 1$ individuals that do not keep their endowment.

Now towards a contradiction, assume that there are $4k_v + 2$ individuals that do not keep their endowment. Then if this additional individual is b_{v,w_i} it must be that $c_{v,w_{i+1}}$ or g_{v,w_i} also does not keep their endowment, a contradiction as it leads to more than $4k_v + 2$ individuals not keeping their endowment. A similar contradiction can be reached for all individuals in b_v, c_v, d_v, e_v and g_v. □

Now, assume that V' is not a vertex cover. Then there is an edge (v, w) such that $v, w \notin V'$. This means that both $a_{v,1}$ and $a_{w,1}$ keep their endowment. From the previous lemma, this implies that we have the following (long) cycle among individuals that keep their endowment.

$$a_{v,1}, b_{v,w_1}, \ldots, b_{v,w}, g_{v,w,1}, \ldots, g_{v,w,Z}, d_{w,v}, e_{w,v}, \ldots, a_{w,1}, \ldots, b_{w,v}, g_{w,v,1}, \ldots g_{w,v,Z}, b_{v,w}, \ldots, a_{v,1}.$$

This gives a contradiction with Pareto optimality.

□

A.3 Proof of Theorem 3

Proof. Assume that $(x(i,h), u(i), v(i))_{i\in I, h\in H}$ is a solution to MINDIST-ILP. Let us show that the matching μ, where $\mu(i) = h$ iff $x(i,h) = 1$ solves MINDIST. First of all, the constraints IP-1 and IP-2 guarantee that μ is indeed a matching. Also notice that $\sum_{i\in I} x(i, \omega(i)) = |I| - d(\mu, \omega)$ so maximising the objective is the same as minimising the distance. So the only thing left to show is that μ is IR and PE.

Towards a contradiction assume that either IR or PE is violated. For the first, assume that IR is not satisfied. Hence, $x(i,h) = 1$ and $\omega(i)\ P_i\ h$. So, for every $h \in H$, $x(i,h)r_i(h, \omega(i)) = 0$. It is a contradiction that $(x(i,h), u(i), v(i))_{i\in I, h\in H}$ is a solution to MINDIST-ILP.

For the second, assume that PE is not satisfied. Then there is a matching μ' such that for all $i \in I$, $\mu'(i)\ R_i\ \mu(i)$ and for at least one $i \in I$, $\mu'(i)\ P_i\ \mu(i)$. Let $s_1 \in I$ be such that $\mu'(s_1)\ P_{s_1}\ \mu(s_1)$. Let s_2 be an agent for which $\mu'(s_1) = \mu(s_2)$ and $\mu(s_2) \neq \mu'(s_2)$. Such agent must exist as otherwise s_1 would not be able to receive $\mu'(s_1)$ in allocation μ'. As $\mu'(s_2) \neq \mu(s_2)$ it must be that $\mu'(s_2)\ P_{s_2}\ \mu(s_2)$. This allows us to find an agent s_3 such that $\mu(s_3) = \mu'(s_2)$ and $\mu(s_3) \neq \mu'(s_3)$. Conclude that $\mu'(s_3)\ P_{s_3}\ \mu(s_3)$. Iterating this further, and as there are a finite number of agents, at some point, we must produce a cycle. As such, this procedure generates a sequence of agents $s_1, \ldots, s_n$ such that for all $i \leq n$, $\mu(s_{i+1})\ P_{s_i}\ \mu(s_i)$ and $\mu(s_1)\ P_{s_n}\ \mu(s_n)$.

Notice that for all these agents $x(s_i, \mu(s_i)) = 1$, $x(s_{i+1}, \mu(s_{i+1})) = 1$ and $pr_i(\mu(s_{i+1}), \mu(s_i)) = 1$. Condition IP-3 then requires that $u(s_i) > u(s_{i+1})$. This gives:

$$u(s_1) > u(s_2) > \ldots > u(s_n) > u(s_1),$$

a contradiction.

Now for the reverse, assume that μ solves MINDIST. We define $x(i,h) = 1$ if and only if $\mu(i) = h$.

Consider a directed graph G with vertices the agents in I and an arrow from $i \in I$ to $j \in I$ if $\mu(j)\ P_i\ \mu(i)$, i.e. i points to j if i prefers the allocation of j over their own allocation. As μ is PE, the graph G should have no cycles. For all i let $N(i)$ be the number of agents that are reachable from i in G either via a direct arrow or via a path of arrows. If there is an arrow from i to j then $N(i) \geq N(j)$. Define $u(i) = \frac{N(i)}{|I|} \in [0, 1]$.

Let us show that this gives a solution to MINDIST-ILP. First of all IP-1 and IP-2 are satisfied as μ is an allocation. If IP-3 is violated, then for all $i \in I$ and $h \in H$, $x(i,h)r_i(h, \omega(i)) = 0$. But this means that there exists $i \in I$ such that $\mu(i) = h$

and i prefers $\omega(i)$ over h, contradicting the assumption that μ satisfies IR.

We show that IP-4 is satisfied. Assume not, then there are $i, j \in I$, $h, k \in H$ such that $x(i,h) = 1, x(j,k) = 1$ and $pr_i(h,k) = 1$, while $u(i) < u(j)$. But this means that $\mu(i) = h$, $\mu(j) = k$ and $k \ P_i \ h$. In the graph G' this means that there is an arrow from i to j, so $u(i) = \frac{N(i)}{|I|} \geq \frac{N(j)}{|I|} = u(j)$, a contradiction.

This shows that $(x(i,h), u(i))_{i \in I, h \in H}$ is feasible for MINDIST-ILP. If $(x(i,h), u(i))_{i \in I, h \in H}$ is not optimal, then there is an other optimal solution $(x'(i,h), u'(i))_{i \in I, h \in H}$ such that $\sum_{i \in I} x'(i, \omega(i)) > \sum_{i \in I} x(i, \omega(i))$. Let $\mu'(i) = h$ iff $x'(i,h) = 1$. But then,

$$d(\mu', \omega) = |I| - \sum_{i \in I} x'(i, \omega(i)) < |I| - \sum_{i \in I} x(i, \omega(i)) = d(\mu, \omega),$$

contradicting the assumption that μ solves MINDIST-IR.

$\square$

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